Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Some Theorems concerning the Centre of Gravity.

By F. Franklin.

Lagrange's two theorems on the centre of gravity may be expressed by the equations

$$M\Sigma m_a (GA)^2 = \Sigma m_a m_b (AB)^2, \qquad (1)$$

$$\Sigma m_a (OA)^2 = \Sigma m_a (GA)^2 + M(OG)^2, \qquad (2)$$

where G is the centre of gravity of the particles of mass m_a, m_b, ... situated at the points A, B, ... and M is the sum of the masses. These theorems admit of extensions, in which, instead of the distances GA, AB, ..., there appear the areas of the triangles GAB, ABC, ...; or the volumes of the tetrahedra $GABC$, $ABCD$, ...; and so on, in space of more than three dimensions. Namely, we have the series of relations

$$\left.\begin{array}{r} M\Sigma m_a (GA)^2 = \Sigma m_a m_b (AB)^2, \\ M\Sigma m_a m_b (GAB)^2 = \Sigma m_a m_b m_c (ABC)^2, \\ M\Sigma m_a m_b m_c (GABC)^2 = \Sigma m_a m_b m_c m_d (ABCD)^2, \\ \text{etc.} \end{array}\right\} \qquad \text{I.}$$

$$\left.\begin{array}{r} \Sigma m_a (OA)^2 = \Sigma m_a (GA)^2 + M(OG)^2, \\ \Sigma m_a m_b (OAB)^2 = \Sigma m_a m_b (GAB)^2 + M\Sigma m_a (OGA)^2, \\ \Sigma m_a m_b m_c (OABC)^2 = \Sigma m_a m_b m_c (GABC)^2 + M\Sigma m_a m_b (OGAB)^2, \\ \text{etc.} \end{array}\right\} \qquad \text{II.}$$

These theorems may be proved almost instantaneously. I shall give the proof for the case of triangles, any other case being obviously demonstrable in the same way.

I. Let the origin of coordinates be the centre of gravity, and consider the matrices

$$\left|\begin{array}{lllll} m_a & m_b & m_c & m_d & \ldots. \\ m_a y_a & m_b y_b & m_c y_c & m_d y_d & \ldots. \\ m_a z_a & m_b z_b & m_c z_c & m_d z_d & \ldots. \end{array}\right|, \quad \left|\begin{array}{lllll} 1 & 1 & 1 & 1 & \ldots. \\ y_a & y_b & y_c & y_d & \ldots. \\ z_a & z_b & z_c & z_d & \ldots. \end{array}\right|,$$

and the similar pairs of matrices in z, x and x, y. The sum of the products of corresponding determinants in these matrices is evidently $4\Sigma m_a m_b m_c (ABC)^2$. On the other hand, multiplying the pair of matrices above written, we get

$$\begin{vmatrix} M, & 0\ , & 0\ , \\ 0\ , & \Sigma m_a y_a^2\ , & \Sigma m_a y_a z_a, \\ 0\ , & \Sigma m_a y_a z_a, & \Sigma m_a z_a^2\ , \end{vmatrix} = M \begin{vmatrix} m_a y_a, & m_b y_b, & m_c y_c, & \ldots \\ m_a z_a\ , & m_b z_b\ , & m_c z_c\ , & \ldots \end{vmatrix} . \begin{vmatrix} y_a, & y_b, & y_c, & \ldots \\ z_a\ , & z_b\ , & z_c\ , & \ldots \end{vmatrix} .$$

Now, since G is the origin, the determinant $\begin{vmatrix} y_a & y_b \\ z_a & z_b \end{vmatrix}$ is twice the projection, on the plane of yz, of the triangle GAB; hence the sum of the products of corresponding determinants in the last-written pair of matrices and in the similar pairs in z, x and x, y is $4M\Sigma m_a m_b (GAB)^2$. Therefore

$$M\Sigma m_a m_b (GAB)^2 = \Sigma m_a m_b m_c (ABC)^2.$$

II. Let the origin be at any point O; let the coordinates of G be x, y, z; let those of A be $x + x_a$, $y + y_a$, $z + z_a$, etc. Then on the one hand

$$\begin{vmatrix} m_a(y + y_a), & m_b(y + y_b), & m_c(y + y_c), & \ldots \\ m_a(z + z_a)\ , & m_b(z + z_b), & m_c(z + z_c), & \ldots \end{vmatrix} . \begin{vmatrix} y + y_a, & y + y_b, & y + y_c, & \ldots \\ z + z_a\ , & z + z_b\ , & z + z_c\ , & \ldots \end{vmatrix} ,$$

added to the two similar products, gives $4\Sigma m_a m_b (OAB)^2$. On the other hand, if we apply the process for the multiplication of matrices, and bear in mind that $\Sigma m_a y_a = 0$, $\Sigma m_a z_a = 0$, it is plain that the product of the above two matrices is equivalent to that of the two following:

$$\begin{vmatrix} m_a y, & m_a y_a, & m_b y, & m_b y_b, & m_c y, & m_c y_c, & \ldots \\ m_a z\ , & m_a z_a\ , & m_b z\ , & m_b z_b\ , & m_c z\ , & m_c z_c\ , & \ldots \end{vmatrix} , \begin{vmatrix} y, & y_a, & y, & y_b, & y, & y_c, & \ldots \\ z\ , & z_a\ , & z\ , & z_b\ , & z\ , & z_c\ , & \ldots \end{vmatrix} .$$

And the product of these, added to the two similar products, evidently gives $4\{\Sigma m_a m_b (GAB)^2 + M\Sigma m_a (OGA)^2\}$. Hence

$$\Sigma m_a m_b (OAB)^2 = \Sigma m_a m_b (GAB)^2 + M\Sigma m_a (OGA)^2.$$

Any theorem under I is a sufficiently obvious corollary of the corresponding theorem under II; but it seemed worth while to give an independent proof of the former. The method of that proof gives rise also to a theorem concerning the principal moments of a system at the centre of gravity which seems not without interest. If, besides taking the origin at the centre of gravity, we take for axes of coordinates, the principal axes of the system of particles, the multiplication of the matrices

$$\begin{vmatrix} m_a\ , & m_b\ , & m_c\ , & m_d\ , & \ldots \\ m_a x_a , & m_b x_b , & m_c x_c , & m_d x_d , & \ldots \\ m_a y_a , & m_b y_b , & m_c y_c , & m_d y_d , & \ldots \\ m_a z_a , & m_b z_b , & m_c z_c , & m_d z_d , & \ldots \end{vmatrix} , \quad \begin{vmatrix} 1\ , & 1\ , & 1\ , & 1\ , & \ldots \\ x_a , & x_b , & x_c , & x_d , & \ldots \\ y_a , & y_b , & y_c , & y_d , & \ldots \\ z_a , & z_b , & z_c , & z_d , & \ldots \end{vmatrix} ,$$

gives the equation

$$M \Sigma m_a x_a^2 . \Sigma m_a y_a^2 . \Sigma m_a z_a^2 = 36 \Sigma m_a m_b m_c m_d (ABCD)^2 .$$

Treating in the same way the similar pairs of matrices with three and two rows respectively, and writing

$$\Sigma m_a x_a^2 = A', \ \Sigma m_a y_a^2 = B', \ \Sigma m_a z_a^2 = C',$$

$$\Sigma m_a m_b (AB)^2 = \Sigma_2, \ 2^2 \Sigma m_a m_b m_c (ABC)^2 = \Sigma_3, \ 6^2 \Sigma m_a m_b m_c m_d (ABCD)^2 = \Sigma_4,$$

we have

$$M(A' + B' + C') = \Sigma_2, \ M(B'C' + C'A' + A'B') = \Sigma_3, \ MA'B'C' = \Sigma_4,$$

so that A', B', C' are the roots of the equation

$$M\lambda^3 - \Sigma_2 \lambda^2 + \Sigma_3 \lambda - \Sigma_4 = 0 .$$

Thus the principal moments at the centre of gravity are determined in terms of the masses and mutual distances of the particles. Of course a similar equation holds for any number of dimensions.

The moments of inertia themselves are $B' + C'$, $C' + A'$, $A' + B'$, and the equation of which these are the roots is

$$M^2 \mu^3 - 2M\Sigma_2 \mu^2 + (\Sigma_2^2 + M\Sigma_3)\mu - (\Sigma_2 \Sigma_3 - M\Sigma_4) = 0 .$$